Journal of Statistical Physics, Vol. 19, No. 5, 1978

Block Spins in the Edge of an Ising Ferromagnetic Half-Plane

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Received January 16, 1978; revised July 11, 1978

The characteristic function of a block spin in the face of an Ising ferromagnetic half-plane is obtained in closed form. The distribution function for the block spin converges to a Gaussian at the critical temperature, but the normalization of the block is modified.

KEY WORDS: Block spins; Ising ferromagnet; half-plane edge; Gaussian distribution.

1. INTRODUCTION

The ideas of Kadanoff[1] on block variables and their relevance in analyzing the critical properties of matter are of central relevance in Wilson theory.[2] Recently some rigorous results on the limit theorems for probability distributions of such variables have been given.[3,4] Bleher and Sinai[5] have analyzed a hierarchical model in considerable detail. Evidently considerable interest attaches to the departure from Gaussian behavior of fluctuations of block variables, suitably normalized. A new result is presented here for a block spin in the face of an Ising ferromagnetic half-plane; the characteristic function is given. It has Gaussian behavior, but the normalization is modified.

2. FORMULATION

Let the spins $\sigma_{\mathbf{i}} = \pm 1$ be located at the vertices $\mathbf{i} = (i_1, i_2)$ of a cylindrical lattice Λ with its axis in the i_2 direction. Here we have $1 \leqslant i_1 \leqslant M$, $1 \leqslant i_2 \leqslant N$. The energy of a configuration $\{\sigma\}$ of spins is given by

$$E_\Lambda(\{\sigma\}) = -J \sum_{\langle \mathbf{i}, \mathbf{j} \rangle} \sigma_{\mathbf{i}} \sigma_{\mathbf{j}} \tag{1}$$

Partially supported by NRC grant A9344.

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0022-4715/78/1100-0553$05.00/0

where $\langle i, j\rangle$ denotes summation over nearest neighbors and $J > 0$ is a ferromagnetic coupling; (1) has the cyclic symmetry of the cylinder. The canonical ensemble is defined by the probability measures

$$p_\Lambda(\{\sigma\}) = Z_\Lambda^{-1} \exp[-\beta E_\Lambda(\{\sigma\})] \tag{2}$$

where $\beta = 1/k_B T$; hereafter $K = \beta J$. In this paper we calculate

$$C_M(\theta) = \lim_{N\to\infty} \left\langle \exp\left(i\theta \sum_1^M \sigma_{j1}\right)\right\rangle \tag{3}$$

where $\langle\cdots\rangle$ denotes expectation with respect to (2). Our procedure uses the transfer matrix[6,7] and ghost-spin technique. We consider a zeroth row of spins $\mathscr{R}_0$ coupled to $\mathscr{R}_1$ by nearest-neighbor interactions of strength $i\theta$. Then the spins in $\mathscr{R}_0$ are constrained to point up. Using the notation

$$V_1(y) = \exp\left(-y \sum_1^M \sigma_j^z\right) \tag{4}$$

$$V_2(y) = \exp\left(y \sum_1^M \sigma_j^x \sigma_{j+1}^x\right), \qquad \sigma_{M+1}^x = \sigma_1^x \tag{5}$$

and

$$\sigma_j^x|0\rangle = |+\rangle \tag{6}$$

$$\sigma_j^z|0\rangle = -|0\rangle, \qquad j = 1,\dots, M \tag{7}$$

we have

$$C_M(\theta) = \lim_{N\to\infty} \frac{(2i\sin 2\theta)^{M/2}\langle + | V_1((K^* - \theta^*)/2)(V')^N V_1(-K^*/2)|0\rangle}{\langle + |0\rangle\langle 0| V_1(-K^*/2)(V')^N V_1(-K^*/2)|0\rangle} \tag{8}$$

where

$$\exp 2K^* = \coth K, \qquad \exp 2\theta^* = -i\cot\theta \tag{9}$$

For any finite M, V' has a unique maximal eigenvector, denoted $|\Phi_+\rangle$. The limit $N \to \infty$ is easily taken, to give

$$C_M(\theta) = \frac{(2ie^{K^*}\sin 2\theta)^{M/2}\langle + | V_1((K^* - \theta^*)/2)|\Phi_+\rangle}{\langle + |0\rangle\langle 0|\Phi_+\rangle} \tag{10}$$

By using the reduction techniques of Refs. 6 and 8, it is easily shown that

$$C_M(\theta) = \prod_{>0}^{<\pi} \{1 - \tan[\delta'(\beta)/2]\cot(\beta/2)\exp(2K^* - 4\theta^*)\} \tag{11}$$

where $\exp(i\beta M) = -1$ in the product and

$$e^{i\delta'(\omega)} = \left[\frac{(e^{i\omega} - A)(e^{i\omega} - B)}{(e^{i\omega} - A^{-1})(e^{i\omega} - B^{-1})}\right]^{1/2} \frac{1}{(AB)^{1/2}} \tag{12}$$

with $A = (\coth K)e^{2K}$ and $B = (\tanh K)e^{2K}$.

The question to which we should like to address ourselves is this: do the block spins

$$\tau_M = M^{-\rho} \sum_1^M \sigma_{m1} \tag{13}$$

have a limiting distribution law as $M \to \infty$? Outside the critical region, with $\rho = 1$ so that $0 < \langle \tau_M{}^2 \rangle < \infty \; \forall M$, one would anticipate a Gaussian[3,4]; putting $\theta = \varphi/M$ in (11) gives

$$G(\varphi) = \lim_{M\to\infty} C_M(\varphi/M) = \exp(-\alpha\varphi^2) \tag{14}$$

where

$$\alpha = (1/2\pi) \int_0^{2\pi} d\omega \tan[\delta'(\omega)/2] \cot(\omega/2) \tag{15}$$

Note that as $K \to K_c-$, where $\sinh 2K_c = 1$ gives the critical temperature, $\alpha \sim \log(K_c - K)/K_c$. Thus we may anticipate that the distribution will behave singularly at $K = K_c$. Setting $K = K_c$, it is clear that $0 < \langle \tau_M{}^2 \rangle < \infty$ only if ρ is M dependent: in fact,

$$\rho_M = 1 + \log\log M/\log M \tag{16}$$

Bearing in mind limit theorem results for independent random variables,[3] we only anticipate sensible results if $\theta = \varphi/M^{\rho_M}$. In this case a careful analysis of the product (11) gives

$$-\log G(\varphi)$$
$$= \lim_{M\to\infty} \frac{\varphi^2}{M \log M} \sum_{\beta>0}^{<\pi} \cot\frac{\beta}{2} \tan\frac{\delta_c'(\beta)}{2}$$
$$= \varphi^2 \lim_{M\to\infty} \frac{1}{M\log M} \left\{ \sum_{\beta>0}^{<\pi} \left[\cot\left(\frac{\beta}{2}\right) - \frac{2}{\beta}\right] \tan\frac{\delta_c'}{2} + 2\sum_{>0}^{<\pi} \beta^{-1} \tan\frac{\delta_c'}{2} \right\}$$

Using the boundedness of $\tan[\delta_c'(\beta)/2]$ in β, which is uniform in M, the contribution of the first term in brackets vanishes in the limit. The second term is written

$$2\sum_{>0}^{<\pi} \beta^{-1}\{\tan[\delta_c'(\beta)/2] - 1\} + 2\sum_{>0}^{<\pi} 1/\beta$$

Since $\delta_c'(0) = \pi/2 (\bmod 2\pi)$, the first term grows as M and therefore gives vanishing contribution in the limit. By definition, $\beta = (2r + 1)\pi/M$, $r \in \mathbb{Z}$; the last sum diverges as $\pi^{-1} M \log M$, giving the result

$$G_c(\varphi) = \exp(-\varphi^2/\pi) \tag{17}$$

The characteristic function for two block spins in the edge can also be obtained; this will be published elsewhere.

The reader familiar with the work of McCoy and Wu [9] should note that the results reported here differ markedly from theirs because of the coupling of limits implied in (14) and (17).

ACKNOWLEDGMENTS

The author wishes to thank Prof. G. Gallavotti for interesting him in this field; he would like to thank him and Prof. M. L. Glasser for a number of interesting discussions, and to acknowledge the hospitality of Nijmegen and Waterloo Universities, where the bulk of this work was done.

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